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# Solvability for an impulsive fractional multi-point boundary value problem at resonance

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## Abstract

In this paper, we consider the fractional multi-point boundary value problem with impulse effects. As indicated by M Fečkan, Y Zhou and JR Wang in 2012, the concepts of piecewise continuous solutions used in most of the current literature about the impulsive differential equations of fractional order are not appropriate. Based on a new concept of a piecewise continuous solution, we establish sufficient conditions for the existence of the solutions for the boundary value problem at resonance by the continuation theorem of coincidence degree theory.

**MSC:** 26A33; 34A08; 34A37**Keywords:** fractional differential equations; impulsive differential equations; coincidence degree; at resonance

## 1 Introduction

In this paper, we are concerned with the existence of the impulsive differential equation of fractional order

$$\begin{cases} {}^C D_{0+}^{\alpha} u(t) = f(t, u(t), u'(t)), & t \in J', \\ \Delta u(t_k) = I_k(u(t_k^-), u'(t_k^-)), & k = 1, 2, \dots, p, \\ \Delta u'(t_k) = J_k(u(t_k^-), u'(t_k^-)), & k = 1, 2, \dots, p, \\ u(0) = 0, \quad u'(1) = \sum_{j=1}^{m-2} \sigma_j u'(\xi_j), \end{cases} \quad (1.1)$$

where  ${}^C D_{0+}^{\alpha}$  is the Caputo fractional derivative,  $1 < \alpha \leq 2$ ,  $f : [0, 1] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  is continuous,  $I_k, J_k : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  ( $k = 1, 2, \dots, p$ ) are continuous,  $0 = t_0 < t_1 < t_2 < \dots < t_p < t_{p+1} = 1$ ,  $J = [0, 1]$ ,  $J' = J \setminus \{t_1, t_2, \dots, t_p\}$ ,  $\sigma_j \in \mathbb{R}$ ,  $\xi_j \in (0, 1)$ ,  $j = 1, 2, \dots, m-2$ ,  $m, p \in \mathbb{N}$ ,  $m > 2$ ,  $p > 0$ , and  $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$ , where  $u(t_k^+)$  and  $u(t_k^-)$  denote the right and left limit of  $u(t)$  at  $t = t_k$  ( $k = 1, 2, \dots, p$ ), respectively.  $\Delta u'(t_k)$  has a similar meaning for  $u'(t)$ . Further, we assume that

$$\xi_i \neq t_k, \quad k = 1, 2, \dots, p, i = 1, 2, \dots, m-2 \quad (1.2)$$

and

$$\sum_{j=1}^{m-2} \sigma_j = 1, \quad (1.3)$$

which implies that the problem (1.1) is at resonance. The problem (1.1) happens to be at resonance in the sense that the kernel of the linear operator  ${}^C D_{0+}^\alpha$  is not less than one-dimensional under the boundary value conditions.

Fractional calculus is a generalization of the ordinary differentiation and integration. It has played a significant role in science, engineering, economy, and other fields. Some books on fractional calculus and fractional differential equations have appeared recently (see [1–3]), and there are a large number of papers dealing with the fractional differential equations (see [4–17]) due to their various applications.

Impulsive differential equations have found its importance in realistic mathematical modeling of the phenomena in both the physical and the social sciences. The boundary value problem of impulsive differential equation has been investigated extensively in the literature; see [9, 10, 12, 15, 18, 19] and references therein.

There are some papers considering the fractional impulsive differential equations today. However, in [9, 15], the authors indicated that the concept of piecewise continuous solutions used in most of the current literature about the impulsive differential equations of fractional order are not appropriate.

Motivated by the papers [6, 8, 9, 11, 15, 17, 20], in this paper we deal with the problem (1.1). Based on the new concept of a piecewise continuous solution presented in [9, 15], we establish some existence results about the problem (1.1). As far as we know, there are few papers to deal with fractional differential equations with impulses under resonant conditions.

The rest of the paper is organized as follows. In Section 2, we introduce some notations, definitions, and preliminary facts that will be used in the remainder of the paper. In Section 3, applying the results listed in Section 2, we prove the existence of the solution for the problem (1.1) by the coincidence degree theory. Then an example is given in Section 4 to demonstrate the application of our results.

## 2 Preliminaries

First of all, we present the necessary definitions and fundamental facts on the fractional calculus theory. These can be found in [1, 3].

**Definition 2.1** ([1, 3]) The Riemann-Liouville fractional integral of order  $\nu > 0$  of a function  $h : [0, 1] \rightarrow \mathbb{R}$  is given by

$$I_{0+}^\nu h(t) = D_{0+}^{-\nu} h(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} h(s) ds \quad (2.1)$$

provided that the right-hand side is pointwise defined on  $[0, 1]$ .

**Definition 2.2** ([1, 3]) The Riemann-Liouville fractional derivative of order  $\nu > 0$  of a continuous function  $h : [0, 1] \rightarrow \mathbb{R}$  is given by

$$D_{0+}^\nu h(t) = \frac{1}{\Gamma(n-\nu)} \left( \frac{d}{dt} \right)^n \int_0^t (t-s)^{n-\nu-1} h(s) ds, \quad (2.2)$$

where  $n = [\nu] + 1$ ,  $h \in AC^n[0, 1]$ , provided that the right-hand side is pointwise defined on  $[0, 1]$ .

**Definition 2.3** ([1, 3]) The Caputo fractional derivative of order  $\nu > 0$  of a function  $h : [0, 1] \rightarrow \mathbb{R}$  is given by

$${}^C D_{0+}^{\nu} h(t) = \frac{1}{\Gamma(n-\nu)} \int_0^t (t-s)^{n-\nu-1} h^{(n)}(s) ds, \quad (2.3)$$

where  $n = [\nu] + 1$ ,  $h \in AC^n[0, 1]$ , provided that the right-hand side is pointwise defined on  $[0, 1]$ .

**Lemma 2.1** ([16]) Let  $\nu > 0$ ; then the differential equation

$${}^C D_{0+}^{\nu} h(t) = 0$$

has solutions  $h(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$ ,  $c_i \in \mathbb{R}$ ,  $i = 0, 1, 2, \dots, n-1$ ,  $n = [\nu] + 1$ .

**Lemma 2.2** ([16]) Let  $\nu > 0$ ; then

$$I_{0+}^{\nu} {}^C D_{0+}^{\nu} h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},$$

for some  $c_i \in \mathbb{R}$ ,  $i = 0, 1, 2, \dots, n-1$ , where  $n = [\nu] + 1$ .

**Lemma 2.3** ([1, 3]) If  $\nu_1, \nu_2, \nu > 0$ ,  $t \in [0, 1]$ , and  $h(t) \in L^1[0, 1]$ , then we have

$$I_{0+}^{\nu_1} I_{0+}^{\nu_2} h(t) = I_{0+}^{\nu_1+\nu_2} h(t), \quad {}^C D_{0+}^{\nu} I_{0+}^{\nu} h(t) = h(t). \quad (2.4)$$

Now let us recall some notations as regards the coincidence degree continuation theorem.

Let  $X, Z$  be real Banach spaces. Consider the operation equation

$$Lu = Nu,$$

where  $L : \text{dom } L \subset X \rightarrow Z$  is a linear operator,  $N : X \rightarrow Z$  is a nonlinear operator. If  $\dim \text{Ker } L = \text{codim Im } L < +\infty$  and  $\text{Im } L$  is closed in  $Z$ , then  $L$  is called a Fredholm mapping of index zero. If  $L$  is a Fredholm mapping of index zero, there exist linear continuous projectors  $P : X \rightarrow X$  and  $Q : Z \rightarrow Z$  such that  $\text{Ker } L = \text{Im } P$ ,  $\text{Im } L = \text{Ker } Q$ , and  $X = \text{Ker } L \oplus \text{Ker } P$ ,  $Z = \text{Im } L \oplus \text{Im } Q$ . Then it follows that  $L_P = L|_{\text{dom } L \cap \text{Ker } P} : \text{dom } L \cap \text{Ker } P \rightarrow \text{Im } L$  is invertible. We denote the inverse of this map by  $K_P$ . If  $\overline{\Omega}$  is an open bounded subset of  $X$ , the map  $N$  will be called  $L$ -compact on  $\overline{\Omega}$  if  $QN(\overline{\Omega})$  is bounded and  $K_{P,Q}N = K_P(I - Q)N : \overline{\Omega} \rightarrow X$  is compact. For  $\text{Im } Q$  is isomorphic to  $\text{Ker } L$ , there exists an isomorphism  $J_{NL} : \text{Im } Q \rightarrow \text{Ker } L$ . Then we shall give the coincidence degree continuation theorem, which is proved in [21].

**Theorem 2.1** Let  $L$  be a Fredholm operator of index zero and  $N$  be  $L$ -compact on  $\overline{\Omega}$ , where  $\Omega$  is an open bounded subset of  $X$ . Suppose that the following conditions are satisfied:

- (i)  $Lx \neq \lambda Nx$  for each  $(x, \lambda) \in [(\text{dom } L \setminus \text{Ker } L) \cap \partial\Omega] \times (0, 1)$ ;
- (ii)  $Nx \notin \text{Im } L$  for each  $x \in \text{Ker } L \cap \partial\Omega$ ;
- (iii)  $\deg(J_{NL}QN|_{\text{Ker } L}, \Omega \cap \text{Ker } L, 0) \neq 0$ , where  $Q : Z \rightarrow Z$  is a continuous projection as above with  $\text{Im } L = \text{Ker } Q$  and  $J_{NL} : \text{Im } Q \rightarrow \text{Ker } L$  is any isomorphism.

Then the equation  $Lx = Nx$  has at least one solution in  $\text{dom } L \cap \overline{\Omega}$ .

Set  $J_0 = [0, t_1]$  and  $J_k = (t_k, t_{k+1}]$ ,  $k = 1, 2, \dots, p$ . Then we shall introduce some function spaces in the following:

$$PC(J, \mathbb{R}) = \{x \mid x: J \rightarrow \mathbb{R}, x \in C(J_k) \text{ and } x(t_k^+) \text{ exists, } k = 0, 1, 2, \dots, p\},$$

equipped with the norm

$$\|x\|_0 = \sup_{t \in J} |x(t)|$$

and

$$PC^1(J, \mathbb{R}) = \{x \mid x \in PC(J, \mathbb{R}), x \in C^1(J_k) \text{ and } x'(t_k^+) \text{ exists, } k = 0, 1, 2, \dots, p\},$$

equipped with the norm

$$\|x\|_1 = \sup_{t \in J} |x(t)| + \sup_{t \in J} |x'(t)|.$$

Obviously,  $PC(J, \mathbb{R})$  and  $PC^1(J, \mathbb{R})$  are Banach spaces.

Define  $X = PC^1(J, \mathbb{R})$  and  $Z = PC(J, \mathbb{R}) \times \mathbb{R}^{2p}$  with the norm

$$\|z\|_2 = \sup_{t \in J} |x(t)| + \sum_{i=1}^{2p} |z_i|, \quad \forall z = (x, z_1, z_2, \dots, z_{2p}) \in Z,$$

where  $x \in PC(J, \mathbb{R})$ ,  $(z_1, z_2, \dots, z_{2p}) \in \mathbb{R}^{2p}$ . It is easy to verify that  $(Z, \|\cdot\|_2)$  is a Banach space.

Let

$$\begin{aligned} \text{dom } L &= \left\{ x \mid x \in X, {}^C D_{0+}^\alpha u(t) \in PC(J, \mathbb{R}), u(0) = 0, u'(1) = \sum_{j=1}^{m-2} \sigma_j u'(\xi_j) \right\}, \\ L: \text{dom } L &\rightarrow Z, \quad u \rightarrow ({}^C D_{0+}^\alpha u(t), \Delta u(t_1), \dots, \Delta u(t_p), \Delta u'(t_1), \dots, \Delta u'(t_p)) \end{aligned} \quad (2.5)$$

and

$$N: X \rightarrow Z, \quad u \rightarrow (f(t, u(t), u'(t)), A_1, \dots, A_p, B_1, \dots, B_p), \quad (2.6)$$

where  $A_k = I_k(u(t_k^-), u'(t_k^-))$ ,  $B_k = J_k(u(t_k^-), u'(t_k^-))$ ,  $k = 1, 2, \dots, p$ .

Then the multi-point boundary value problem can be written

$$Lu = Nu, \quad u \in \text{dom } L.$$

At the end of the section we give a method for determining the compactness of a set in  $X$ .

**Lemma 2.4**  *$\mathcal{U}$  is a relatively compact set in  $X$  if and only if the following conditions are satisfied:*

- (a)  $\mathcal{U}$  is uniformly bounded, that is, there exists a constant  $R > 0$ , such that for each  $u \in \mathcal{U}$ ,  $\|u\|_1 \leq R$ .  
 (b)  $\forall \varepsilon > 0$ , there exists  $\delta = \delta(\varepsilon) > 0$ , such that

$$|u(t_1) - u(t_2)| < \varepsilon, \quad |u'(t_1) - u'(t_2)| < \varepsilon,$$

for  $\forall t_1, t_2 \in J_k$ ,  $k = 0, 1, 2, \dots, p$ ,  $|t_1 - t_2| < \delta$ ,  $\forall u \in \mathcal{U}$ .

*Proof* Analogous to the proof of the Lemma 2.2 in [17], we can prove the results. Here, we omit the details.  $\square$

### 3 Main results

In this section, we will establish the existence theorem for the impulsive fractional differential equation (1.1). In order to prove our main results, we need the following lemmas.

**Lemma 3.1** Suppose that  $g \in PC(J, \mathbb{R})$  and  $\sum_{j=1}^{m-2} \sigma_j = 1$ . A function  $u \in X$  is a solution of the impulsive differential equation of fractional order

$$\begin{cases} {}^C D_{0+}^\alpha u(t) = g(t), & t \in J', \\ \Delta u(t_k) = a_k, & k = 1, 2, \dots, p, \\ \Delta u'(t_k) = b_k, & k = 1, 2, \dots, p, \\ u(0) = 0, & u'(1) = \sum_{j=1}^{m-2} \sigma_j u'(\xi_j), \end{cases} \quad (3.1)$$

if and only if

$$u(t) = \sum_{i=1}^k a_i - \sum_{i=1}^k b_i t_i + \left( c + \sum_{i=1}^k b_i \right) t + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) ds, \quad t \in J_k, \quad (3.2)$$

where  $k = 0, 1, 2, \dots, p$  and  $c \in \mathbb{R}$ , and

$$\begin{aligned} & \sum_{i=1}^p b_i - \sum_{j=1}^{m-2} \sigma_j \left( \sum_{t_i < \xi_j} b_i \right) \\ & + \frac{1}{\Gamma(\alpha-1)} \left[ \int_0^1 (1-s)^{\alpha-2} g(s) ds - \sum_{j=1}^{m-2} \sigma_j \left( \int_0^{\xi_j} (\xi_j - s)^{\alpha-2} g(s) ds \right) \right] = 0. \end{aligned} \quad (3.3)$$

*Proof* In view of Lemma 2.7 in [9] or Lemma 4.1 in [15], we can get the conclusions by applying Lemma 2.2 and Lemma 2.3.  $\square$

**Lemma 3.2** Assume that  $\sum_{j=1}^{m-2} \sigma_j = 1$  and

$$\Lambda_0 := \frac{1}{\Gamma(\alpha)} \left( 1 - \sum_{j=1}^{m-2} \sigma_j \xi_j^{\alpha-1} \right) \neq 0.$$

Then  $L$  is a Fredholm mapping of index zero. Moreover,

$$\text{Ker } L = \{u \mid u = ct, c \in \mathbb{R}\} \subset X \quad (3.4)$$

and

$$\begin{aligned} \operatorname{Im} L &= \{z \mid z = (g, a_1, \dots, a_p, b_1, \dots, b_p) \in Z \text{ and } g \text{ satisfies the condition (3.3)}\} \\ &\subset Z. \end{aligned} \quad (3.5)$$

*Proof* Taking into account Lemma 2.1 and the definition of  $L$ , we can get the result (3.4) easily. Further, Lemma 3.1 implies (3.5). Now, let us focus on the proof that  $L$  is a Fredholm mapping of index zero.

Define an auxiliary operator  $Q_1 : Z \rightarrow \mathbb{R}$  as follows:

$$\begin{aligned} Q_1 z &= \sum_{i=1}^p b_i - \sum_{j=1}^{m-2} \sigma_j \left( \sum_{t_i < \xi_j} b_i \right) \\ &\quad + \frac{1}{\Gamma(\alpha-1)} \left[ \int_0^1 (1-s)^{\alpha-2} g(s) ds - \sum_{j=1}^{m-2} \sigma_j \left( \int_0^{\xi_j} (\xi_j - s)^{\alpha-2} g(s) ds \right) \right], \end{aligned}$$

where  $z = (g, a_1, \dots, a_p, b_1, \dots, b_p) \in Z$ . It is obvious that  $Q_1$  is a continuous linear mapping.

Take the mapping  $Q : Z \rightarrow Z$  defined by

$$Qz = \left( \frac{Q_1 z}{\Lambda_0}, 0, \dots, 0 \right), \quad (3.6)$$

where  $z = (g, a_1, \dots, a_p, b_1, \dots, b_p) \in Z$  and  $(0, \dots, 0) \in \mathbb{R}^{2p}$ . Evidently,  $\operatorname{Ker} Q = \operatorname{Im} L$  and

$$\operatorname{Im} Q = \{ (c, 0, \dots, 0) \in Z \mid c \in \mathbb{R}, (0, \dots, 0) \in \mathbb{R}^{2p} \},$$

and  $Q : Z \rightarrow Z$  is a continuous linear projector. In fact, for an arbitrary  $z \in Z$ , we have

$$Q_1(Qz) = Q_1 \left( \frac{Q_1 z}{\Lambda_0}, 0, \dots, 0 \right) = \frac{Q_1 z}{\Lambda_0} \Lambda_0 = Q_1 z$$

and

$$Q^2 z = Q(Qz) = \left( \frac{Q_1(Qz)}{\Lambda_0}, 0, \dots, 0 \right) = \left( \frac{Q_1 z}{\Lambda_0}, 0, \dots, 0 \right) = Qz,$$

that is to say,  $Q : Z \rightarrow Z$  is idempotent.

Let  $z = z - Qz + Qz = (I - Q)z + Qz$ , where  $z$  is an arbitrary element in  $Z$ . Since  $Qz \in \operatorname{Im} Q$  and  $(I - Q)z \in \operatorname{Ker} Q$ , we obtain  $Z = \operatorname{Im} Q + \operatorname{Ker} Q$ . Take  $z_0 \in \operatorname{Im} Q \cap \operatorname{Ker} Q$ . Then  $z_0$  can be written as  $z_0 = (c_0, 0, \dots, 0)$ , where  $c_0 \in \mathbb{R}$ ,  $(0, \dots, 0) \in \mathbb{R}^{2p}$ , for  $z_0 \in \operatorname{Im} Q$ . Thus we have  $Q_1(z_0) = c_0 \Lambda_0 = 0$ , which implies that  $c_0 = 0$  and  $z_0 = \theta$ . Hence  $\operatorname{Im} Q \cap \operatorname{Ker} Q = \{\theta\}$ , where we denote  $\theta$  the zero element in  $Z$ . Then we get  $Z = \operatorname{Im} Q \oplus \operatorname{Ker} Q = \operatorname{Im} Q \oplus \operatorname{Im} L$ .

Now,  $\dim \operatorname{Ker} L = 1 = \dim \operatorname{Im} Q = \operatorname{codim} \operatorname{Ker} Q = \operatorname{codim} \operatorname{Im} L < +\infty$ , and noting that  $\operatorname{Im} L$  is closed in  $Z$ ,  $L$  is a Fredholm mapping of index zero.  $\square$

Let  $P : X \rightarrow X$  be defined by

$$(Pu)(t) = u'(0)t, \quad t \in J, u \in X. \quad (3.7)$$

Clearly,  $P: X \rightarrow X$  is a linear continuous projector and

$$\operatorname{Im} P = \{u \mid u = ct, c \in \mathbb{R}\} = \operatorname{Ker} L.$$

Also, proceeding as in the proof of Lemma 3.2, we can show that  $X = \operatorname{Im} P \oplus \operatorname{Ker} P = \operatorname{Ker} L \oplus \operatorname{Ker} P$ .

Consider the mapping  $K_P: \operatorname{Im} L \rightarrow \operatorname{dom} L \cap \operatorname{Ker} P$ :

$$(K_P z)(t) = \sum_{i=1}^k a_i - \sum_{i=1}^k b_i t_i + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) ds, \quad t \in J_k,$$

where  $k = 0, 1, 2, \dots, p$  and  $z = (g, a_1, \dots, a_p, b_1, \dots, b_p) \in \operatorname{Im} L$ .

Note that

$$(K_P L)u = K_P(Lu) = u, \quad \forall u \in \operatorname{dom} L \cap \operatorname{Ker} P \quad (3.8)$$

and

$$(LK_P)z = L(K_P z) = z, \quad \forall z \in \operatorname{Im} L.$$

Thus,  $K_P = (L_P)^{-1}$ , where  $L_P = L|_{\operatorname{dom} L \cap \operatorname{Ker} P}: \operatorname{dom} L \cap \operatorname{Ker} P \rightarrow \operatorname{Im} L$ .

**Lemma 3.3** For each  $z = (g, a_1, \dots, a_p, b_1, \dots, b_p) \in \operatorname{Im} L$ , we have

$$\|K_P z\|_1 \leq \left(1 + \frac{1}{\Gamma(\alpha)}\right) \|z\|_2. \quad (3.9)$$

*Proof* For each  $z = (g, a_1, \dots, a_p, b_1, \dots, b_p) \in \operatorname{Im} L$  and  $t \in J_k$ ,  $k = 0, 1, 2, \dots, p$ , we have

$$\begin{aligned} |(K_P z)(t)| &= \left| \sum_{i=1}^k a_i - \sum_{i=1}^k b_i t_i + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) ds \right| \\ &\leq \sup_{t \in J} |g(t)| \cdot \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \right| + \sum_{i=1}^k |a_i| + \sum_{i=1}^k |b_i| \\ &= \sup_{t \in J} |g(t)| \cdot \frac{1}{\Gamma(\alpha+1)} t^\alpha + \sum_{i=1}^k |a_i| + \sum_{i=1}^k |b_i| \\ &\leq \sup_{t \in J} |g(t)| + \sum_{i=1}^p |b_i| + \sum_{i=1}^p |a_i| = \|z\|_2 \end{aligned}$$

and

$$|(K_P z)'(t)| = \left| \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} g(s) ds \right| \leq \sup_{t \in J} |g(t)| \cdot \frac{1}{\Gamma(\alpha)} t^{\alpha-1} \leq \frac{1}{\Gamma(\alpha)} \|z\|_2.$$

Hence,

$$\|K_P z\|_1 = \sup_{t \in J} |(K_P z)(t)| + \sup_{t \in J} |(K_P z)'(t)| \leq \left(1 + \frac{1}{\Gamma(\alpha)}\right) \|z\|_2. \quad \square$$

**Lemma 3.4** *Let  $f : [0, 1] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  and  $I_k, J_k : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  ( $k = 1, 2, \dots, p$ ) are continuous. Then  $K_{p,Q}N = K_p(I - Q)N : X \rightarrow X$  is completely continuous.*

*Proof* By virtue of Lemma 2.4, we can conclude that the claim of the lemma is true.  $\square$

Next, let us list the assumptions that will be used in the rest of the section.

(H<sub>1</sub>)  $f : [0, 1] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  is continuous, and there exist three nonnegative functions  $\beta_1, \beta_2, \beta_3 \in PC(J, \mathbb{R})$  such that for all  $t \in J$  and  $(x, y) \in \mathbb{R}^2$ , we have

$$|f(t, x, y)| \leq \beta_1(t)|x| + \beta_2(t)|y| + \beta_3(t). \quad (3.10)$$

(H<sub>2</sub>) There exist  $M_i \geq 0, N_i \geq 0, D_i \geq 0$  ( $i = 1, 2, \dots, 2p$ ) such that

$$|I_k(x, y)| \leq M_k|x| + N_k|y| + D_k, \quad |J_k(x, y)| \leq M_{p+k}|x| + N_{p+k}|y| + D_{p+k}, \quad (3.11)$$

where  $k = 1, 2, \dots, p, \forall (x, y) \in \mathbb{R}^2$ .

(H<sub>3</sub>)

$$0 \leq \left(1 + \frac{1}{\Gamma(\alpha)}\right) \left(\eta_1 + \sum_{i=1}^{2p} (M_i + N_i)\right) < 1, \quad (3.12)$$

where  $\eta_1 = \sup_{t \in J} |\beta_1(t)| + \sup_{t \in J} |\beta_2(t)|$ .

(H<sub>4</sub>) There exists a constant  $T_1 > 0$  such that

$$Q_1(Nu) \neq 0, \quad (3.13)$$

for each  $u \in \text{dom } L \setminus \text{Ker } L$  satisfying  $|u'(0)| > T_1$ .

(H<sub>5</sub>) There exists a positive constant  $S$  such that for any  $c \in \mathbb{R}$ , if  $|c| > S$ , then either

$$c\Lambda_0 Q_1(N(ct)) < 0, \quad (3.14)$$

or else

$$c\Lambda_0 Q_1(N(ct)) > 0. \quad (3.15)$$

**Theorem 3.1** *Let  $\sum_{j=1}^{m-2} \sigma_j = 1, \Lambda_0 \neq 0, f : [0, 1] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  and  $I_k, J_k : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  ( $k = 1, 2, \dots, p$ ) are continuous. Assume that the hypotheses (H<sub>1</sub>)-(H<sub>5</sub>) all hold. Then the problem (1.1) has at least one solution in  $\text{dom } L$ .*

*Proof* The proof consists of four main steps as follows.

- (1) Set  $\Omega_1 = \{u \in \text{dom } L \setminus \text{Ker } L \mid Lu = \lambda Nu, \lambda \in [0, 1]\}$  and proof of  $\Omega_1$  is bounded.
- (2) Set  $\Omega_2 = \{u \in \text{Ker } L \mid Nu \in \text{Im } L\}$  and proof of  $\Omega_2$  is bounded.
- (3) If (3.14) holds, we set  $\Omega_3 = \{u \in \text{Ker } L \mid -\lambda u + (1 - \lambda)J_{NL}QNu = 0, \lambda \in [0, 1]\}$ ; if (3.15) holds, we set  $\Omega_3 = \{u \in \text{Ker } L \mid \lambda u + (1 - \lambda)J_{NL}QNu = 0, \lambda \in [0, 1]\}$ , where  $J_{NL} : \text{Im } Q \rightarrow \text{Ker } L$  is a linear isomorphism defined as

$$J_{NL}(c, 0, \dots, 0) = ct, \quad (c, 0, \dots, 0) \in \text{Im } Q, t \in [0, 1]. \quad (3.16)$$

Then proof of  $\Omega_3$  is bounded.



(4) Let  $\Omega$  be a bounded open set such that  $\Omega \supset \bigcup_{i=1}^3 \overline{\Omega}_i$  and prove that

$$\deg(J_{NL}QN|_{\text{Ker } L}, \Omega \cap \text{Ker } L, 0) \neq 0.$$

Now, let us prove the steps one by one.

Step 1: Take  $u \in \Omega_1$ , then  $u \in \text{dom } L \setminus \text{Ker } L$  and  $Lu = \lambda Nu$ , so  $\lambda \neq 0$  and  $Nu \in \text{Im } L = \text{Ker } Q \subset Z$ . Hence,  $Q(Nu) = \theta$ , that is,  $Q_1(Nu) = 0$ . From  $(H_4)$ , we have  $|u'(0)| \leq T_1$ .

Again, for  $u \in \Omega_1$ , we get

$$\begin{aligned} \|Pu\|_1 &= \sup_{t \in J} |Pu(t)| + \sup_{t \in J} |(Pu)'(t)| \\ &= \sup_{t \in J} |u'(0)t| + \sup_{t \in J} |u'(0)| = 2|u'(0)| \leq 2T_1. \end{aligned} \quad (3.17)$$

In view of  $(I - P)u \in \text{dom } L \cap \text{Ker } P$ , by (3.8) and Lemma 3.3, we have

$$\begin{aligned} \|(I - P)u\|_1 &= \|K_p L(I - P)u\|_1 \leq \left(1 + \frac{1}{\Gamma(\alpha)}\right) \|L(I - P)u\|_2 \\ &= \left(1 + \frac{1}{\Gamma(\alpha)}\right) \|Lu\|_2 \leq \left(1 + \frac{1}{\Gamma(\alpha)}\right) \|Nu\|_2. \end{aligned} \quad (3.18)$$

Combining (3.17) and (3.18), we can obtain

$$\begin{aligned} \|u\|_1 &= \|u - Pu + Pu\|_1 \leq \|Pu\|_1 + \|(I - P)u\|_1 \\ &\leq 2T_1 + \left(1 + \frac{1}{\Gamma(\alpha)}\right) \|Nu\|_2. \end{aligned} \quad (3.19)$$

From  $(H_1)$ , for each  $u \in \Omega_1$ , we have

$$\begin{aligned} |f(t, u(t), u'(t))| &\leq \beta_1(t)|u(t)| + \beta_2(t)|u'(t)| + \beta_3(t) \\ &\leq \sup_{t \in J} |\beta_1(t)| \cdot |u(t)| + \sup_{t \in J} |\beta_2(t)| \cdot |u'(t)| + \sup_{t \in J} |\beta_3(t)| \\ &\leq \eta_1 \|u\|_1 + \eta_2, \end{aligned} \quad (3.20)$$

where we denote  $\eta_2 = \sup_{t \in J} |\beta_3(t)|$ .

From  $(H_2)$ , for each  $u \in \Omega_1$ , we get

$$\begin{aligned} |I_k(u(t_k^-), u'(t_k^-))| &\leq M_k |u(t_k^-)| + N_k |u'(t_k^-)| + D_k \\ &\leq (M_k + N_k) \|u\|_1 + D_k \end{aligned} \quad (3.21)$$

and

$$\begin{aligned} |J_k(u(t_k^-), u'(t_k^-))| &\leq M_{p+k} |u(t_k^-)| + N_{p+k} |u'(t_k^-)| + D_{p+k} \\ &\leq (M_{p+k} + N_{p+k}) \|u\|_1 + D_{p+k}, \end{aligned} \quad (3.22)$$

where  $k = 1, 2, \dots, p$ .

Combining (3.20), (3.21), and (3.22), we obtain

$$\begin{aligned}\|Nu\|_2 &= \sup_{t \in J} |f(t, u(t), u'(t))| + \sum_{i=1}^p |I_k(u(t_k^-), u'(t_k^-))| + \sum_{i=1}^p |J_k(u(t_k^-), u'(t_k^-))| \\ &\leq \eta_1 \|u\|_1 + \eta_2 + \left( \sum_{i=1}^{2p} (M_i + N_i) \right) \|u\|_1 + \sum_{i=1}^{2p} D_i \\ &= \left( \eta_1 + \sum_{i=1}^{2p} (M_i + N_i) \right) \|u\|_1 + \sum_{i=1}^{2p} D_i + \eta_2.\end{aligned}\quad (3.23)$$

Thus, by (H<sub>3</sub>), (3.19), and (3.23), we can derive

$$\|u\|_1 \leq \frac{2T_1 + (1 + 1/\Gamma(\alpha)) \sum_{i=1}^{2p} D_i + (1 + 1/\Gamma(\alpha)) \eta_2}{1 - (1 + 1/\Gamma(\alpha))(\eta_1 + \sum_{i=1}^{2p} (M_i + N_i))},$$

which clearly shows that  $\Omega_1$  is bounded.

Step 2: Let  $u \in \Omega_2$ , then  $u \in \text{Ker } L$  and  $u = ct$ ,  $c \in \mathbb{R}$ ,  $t \in J$ . Since  $Nu \in \text{Im } L = \text{Ker } Q$ , we have  $Q(Nu) = \theta$ , that is,  $Q_1(N(ct)) = 0$ . Taking into account (H<sub>5</sub>),  $|c| \leq S$ , which implies that  $\Omega_2$  is bounded.

Step 3: Without loss of generality, in the following part of the proof, we assume that (3.14) holds in (H<sub>5</sub>). Then we set  $\Omega_3 = \{u \in \text{Ker } L \mid -\lambda u + (1 - \lambda)J_{NL}QNu = 0, \lambda \in [0, 1]\}$ . For  $u \in \Omega_3$ , then we have  $u = ct$ ,  $c \in \mathbb{R}$ ,  $t \in [0, 1]$ , and  $\lambda u = (1 - \lambda)J_{NL}QNu$ . Thus,

$$\lambda ct = (1 - \lambda) \frac{Q_1(N(ct))}{\Lambda_0} t.$$

Therefore, via (H<sub>5</sub>) and (3.14), we have  $|c| \leq S$ , which shows that  $\Omega_3$  is bounded.

Step 4: Let  $\Omega$  be a bounded open set such that  $\Omega \supset \bigcup_{i=1}^3 \overline{\Omega}_i$ . The operator  $N$  is  $L$ -compact on  $\overline{\Omega}$  due to the fact that  $QN(\overline{\Omega})$  is bounded and  $K_{P,Q}N = K_P(I - Q)N : \overline{\Omega} \rightarrow X$  is compact by Lemma 3.4. Then by Step 1 and Step 2, we have

- (i)  $Lu \neq \lambda Nu$  for each  $(u, \lambda) \in [(\text{dom } L \setminus \text{Ker } L) \cap \partial\Omega] \times (0, 1)$ ;
- (ii)  $Nu \notin \text{Im } L$  for each  $u \in \text{Ker } L \cap \partial\Omega$ .

Define  $H(u, \lambda) = -\lambda Lu + (1 - \lambda)J_{NL}QNu$ , where  $I$  is the identity operator in  $X$ . According to the arguments in Step 3, we have

$$H(u, \lambda) \neq 0, \quad \forall u \in \text{Ker } L \cap \partial\Omega,$$

and therefore, via the homotopy property of degree, we obtain

$$\begin{aligned}\deg(J_{NL}QN|_{\text{Ker } L}, \Omega \cap \text{Ker } L, 0) &= \deg(H(\cdot, 0), \Omega \cap \text{Ker } L, 0) \\ &= \deg(H(\cdot, 1), \Omega \cap \text{Ker } L, 0) \\ &= \deg(-I, \Omega \cap \text{Ker } L, 0) \\ &= -1,\end{aligned}$$

which verifies the condition (iii) of Theorem 2.1. Then, applying Theorem 2.1, we conclude that the problem (1.1) has at least one solution in  $\text{dom } L \cap \overline{\Omega}$ . The proof is complete.  $\square$

**Corollary 3.1** *If the conditions  $(H_3)$  and  $(H_4)$  are replaced by:*

$(H_3^*)$

$$0 \leq \left(1 + \frac{3}{\Gamma(\alpha)}\right) \left(\eta_1 + \sum_{i=1}^{2p} (M_i + N_i)\right) < 1. \quad (3.24)$$

$(H_4^*)$  *There exists a constant  $T_1^* > 0$  such that*

$$Q_1(Nu) \neq 0, \quad (3.25)$$

*for each  $u \in \text{dom } L \setminus \text{Ker } L$  satisfying  $|u'(t)| > T_1^*$ .*

*Then the problem (1.1) has at least one solution in  $\text{dom } L$ .*

**Theorem 3.2** *Let  $\sum_{j=1}^{m-2} \sigma_j = 1$ ,  $\sigma_j > 0$  ( $j = 1, 2, \dots, m-2$ ),  $f : [0, 1] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  and  $I_k, J_k : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  ( $k = 1, 2, \dots, p$ ) are continuous. Suppose that  $(H_2)$  and the following conditions are satisfied:*

$(H_3')$

$$0 \leq \left(1 + \frac{3}{\Gamma(\alpha)}\right) \left(\sum_{i=1}^{2p} (M_i + N_i)\right) < 1. \quad (3.26)$$

$(H_6)$  *There exists a constant  $T_3 > 0$ , such that for any  $|y| > T_2$ , we have*

$$\gamma_1(t)T_3 \leq f(t, x, y) \leq \gamma_2(t)T_3, \quad (3.27)$$

*where  $\gamma_1(t), \gamma_2(t) \in C[0, 1]$  and  $\gamma_2(t) \geq \gamma_1(t) \geq 0$ .*

$(H_7)$  *For  $\forall (x, y) \in \mathbb{R}^2$ ,  $J_k(x, y) \geq 0$ ,  $k = 1, 2, \dots, p$ .*

$(H_8)$

$$\int_0^1 (1-s)^{\alpha-2} \gamma_1(s) ds - \sum_{j=1}^{m-2} \sigma_j \left( \int_0^{\xi_j} (\xi_j - s)^{\alpha-2} \gamma_2(s) ds \right) > 0. \quad (3.28)$$

*Then the problem (1.1) has at least one solution in  $\text{dom } L$ .*

**Proof** In view of  $\sigma_j > 0$  ( $j = 1, 2, \dots, m-2$ ) and  $(H_7)$ , we have  $\Lambda_0 > 0$ .

First, let us set  $\Omega_1 = \{u \in \text{dom } L \setminus \text{Ker } L \mid Lu = \lambda Nu, \lambda \in [0, 1]\}$ . Hence, for each  $u \in \Omega_1$ , we have  $Q(Nu) = \theta$ , that is,  $Q_1(Nu) = 0$ . We claim that there exists  $t_1 \in [0, 1]$  such that  $|u'(t_1)| \leq T_2$ ,  $t \in J$ .

In fact, if we assume that  $|u'(t)| > T_2$ , in view of  $\sigma_j > 0$ ,  $(H_6)$ , and  $(H_8)$ , we can get  $Q_1(Nu) > 0$ , which contradicts  $Q_1(Nu) = 0$ . Thus, the claim that there exists  $t_1 \in [0, 1]$  such that  $|u'(t_1)| \leq T_2$  is true.

If  $t_1 = 0$ ,  $|u'(0)| \leq T_2$ . If  $t_1 > 0$ , we have

$$|u'(0)| \leq T_2 + \frac{1}{\Gamma(\alpha)} \|Nu\|_2.$$

Since

$$u''(t) = {}^C D_{0+}^{2-\alpha} ({}^C D_{0+}^{\alpha} u(t)),$$

we have

$$\begin{aligned} \left| \int_0^{t_1} u''(s) ds \right| &= \left| I_{0+}^1 {}^C D_{0+}^{2-\alpha} ({}^C D_{0+}^{\alpha} u(t_1)) \right| \\ &= \left| I_{0+}^{\alpha-1} (Lu)(t_1) \right| \\ &= \left| \frac{1}{\Gamma(\alpha-1)} \int_0^{t_1} (t_1-s)^{\alpha-2} ({}^C D_{0+}^{\alpha} u(s)) ds \right| \\ &\leq \|Lu\|_2 \cdot \frac{1}{\Gamma(\alpha-1)} \int_0^{t_1} (t_1-s)^{\alpha-2} ds \\ &\leq \|Lu\|_2 \cdot \frac{1}{\Gamma(\alpha)} \leq \frac{1}{\Gamma(\alpha)} \|Nu\|_2. \end{aligned}$$

Thus,

$$|u'(0)| = \left| u'(t_1) - \int_0^{t_1} u''(s) ds \right| \leq |u'(t_1)| + \left| \int_0^{t_1} u''(s) ds \right| \leq T_2 + \frac{1}{\Gamma(\alpha)} \|Nu\|_2.$$

Similar to the discussion in Theorem 3.1, we can get

$$\|u\|_1 \leq \frac{2T_2 + T_3(1 + 3/\Gamma(\alpha)) \max_{t \in [0,1]} \gamma_2(t) + (1 + 3/\Gamma(\alpha)) \sum_{i=1}^{2p} D_i}{1 - (1 + 3/\Gamma(\alpha)) (\sum_{i=1}^{2p} (M_i + N_i))}.$$

Therefore,  $\Omega_1$  is bounded.

Also,  $\Lambda_0 > 0$ ,  $(H_6)$ , and  $(H_8)$  imply that  $(H_5)$  is satisfied. Then, proceeding with the proof of the theorem similar to the proof of Theorem 3.1, we can derive the conclusion.  $\square$

## 4 Examples

To illustrate our main results, we shall present an example.

### Example 4.1

$$\begin{cases} {}^C D_{0+}^{1.8} u(t) = f(t, u(t), u'(t)), & t \in J', \\ \Delta u(t_k) = l, & k = 1, 2, 3, \\ \Delta u'(t_k) = J_k, & k = 1, 2, 3, \\ u(0) = 0, & u'(1) = \sum_{j=1}^3 \sigma_j u'(\xi_j), \end{cases} \quad (4.1)$$

where

$$f(t, x, y) = \frac{t^{\alpha-1}}{20} \frac{|y|}{1+|y|} \left( 1 + e^{-|x|} - \frac{\sin^2 t}{1+y^2} \right), \quad t \in [0, 1], (x, y) \in \mathbb{R}^2$$

and

$$J_k(x, y) = \frac{1}{k+1},$$

and  $l \in \mathbb{R}$ ,  $l > 0$ ,  $t_1 = 0.25$ ,  $t_2 = 0.5$ ,  $t_3 = 0.75$ ,  $\xi_1 = 0.2$ ,  $\xi_2 = 0.4$ ,  $\xi_3 = 0.6$ ,  $\sigma_1 = 0.2$ ,  $\sigma_2 = 0.3$ ,  $\sigma_3 = 0.5$ ,  $k_1 = 0$ ,  $k_2 = 1$ ,  $k_3 = 2$ .

From the above conditions, we can find that  $(H_2)$ ,  $(H'_3)$ , and  $(H_7)$  hold. Noting that  $\alpha = 1.8$ ,  $p = 3$ ,  $m = 5$ ,  $\sum_{i=1}^3 \sigma_i = 1$ , for each  $|y| > 2$ , we have

$$\gamma_1(t) \leq f(t, x, y) \leq \gamma_2(t),$$

where  $\gamma_2 = t^{\alpha-1}/10$ , and  $\gamma_1 = t^{\alpha-1}/30$ , so  $(H_6)$  is satisfied. Also we can certify that  $(H_8)$  holds, too.

To sum up the points which we have just indicated, by Theorem 3.2, we can conclude that the problem (4.1) has at least one solution.

#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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